Compactness in metric spaces

Recall from analysis that $A \subseteq \mathbb{R}^n$ is called compact if A is closed and bounded (using the Euclidean metric).

In fact, this definition agrees w/ the topological definition:

Thm: $A \subseteq \mathbb{R}^n$ is compact if and only if it's closed and it's bounded in the euclidean metric d.

Pf: If $A \subseteq \mathbb{R}^n$ is compact, then it's closed since $\mathbb{R}^n$ is Hausdorff.

Cover A by open balls $\{B_r(0) \mid r \in \mathbb{Z}_+\}$. Then there is some finite subcover, so there is some r s.t. $A \subseteq B_r(0) \Rightarrow A$ is bounded.

Now assume A is closed and bounded. Since it's bounded, it's contained in some $B_r(0) \subseteq [-r, r]^n$, which is compact.
So it's a closed subspace of a compact space and is thus compact. $\square$

This theorem depends on the metric, not just the topology induced by the metric. i.e. $\exists$ metric ρ on $\mathbb{R}^n$ that induces the standard topology, but not all closed, ρ-bounded sets are compact. (See HW).

Calculus generalizations

We can generalize two big results from calculus using compact spaces rather than closed intervals:

The extreme value theorem and the uniform continuity theorem.

Thm: (Extreme value theorem) If X is a compact space and $f: X \to \mathbb{R}$ continuous, then f achieves its maximum. i.e. $\exists$ $c \in X$ s.t. $f(x) \le f(c)$ $\forall$ $x \in X$.

Pf: $f(X) \subseteq \mathbb{R}$ is compact, so it is bounded and contains its limit points. Thus $m = \sup f(X) < \infty$.

Thus $\forall \varepsilon > 0$, $(m-\varepsilon, m+\varepsilon) \cap f(X) \neq \emptyset$. Thus $m \in f(X)$ or m is a limit point of $f(X)$, in which case $m \in f(X)$ as well.

Thus, $\exists$ some $c \in X$ s.t. $f(c) = m$. $\square$

Recall the uniform continuity theorem from calculus:

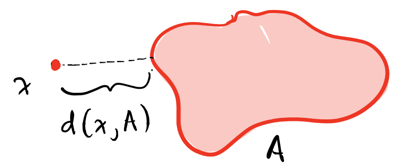
[If $f: [a,b] \to \mathbb{R}$ is continuous, then it is uniformly continuous ($\forall \varepsilon > 0$ $\exists$ $\delta > 0$ s.t. if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$)]

To prove the uniform continuity theorem more generally, we first need a few definitions.

Def: If (X,d) is a metric space and $A \subseteq X$ nonempty

1.) If $x \in X$ the <u>distance</u> from x to A is defined

$$d(x,A) = \inf\{d(x,a) \mid a \in A\}$$



2.) If A is bounded, the <u>diameter</u> of A is

$$\sup\{d(a_1,a_2) \mid a_1, a_2 \in A\}.$$

Exercise: check $d(x,A)$ is a continuous function on X (for fixed A.)

Lemma: Let $\mathcal{A}$ be an open cover of the metric space (X,d). If X is compact $\exists\ \delta > 0$ s.t. $\forall$ subsets having diameter less than δ, there is an element of $\mathcal{A}$ containing it. δ is called a <u>Lebesgue number</u> for $\mathcal{A}$.

Pf: Choose a finite subcover $\{A_1, \dots, A_n\} \subseteq \mathcal{A}$.

Define $f: X \to \mathbb{R}$ by $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, X \setminus A_i)$.

f is a lin. combination of continuous functions and is thus continuous.

Since each A_i is open, if $x \in A_i$, there is some $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A_i$

Thus, $d(x, X \setminus A_i) > \varepsilon$, so $f(x) > 0 \ \forall\ x$.

Since X is compact, it achieves its minimum, so there is some $\delta > 0$ s.t. $f(x) \geq \delta \quad \forall \ x \in X$.

Thus, for each x, there is some A_i s.t. $d(x, X \setminus A_i) \geq \delta$.

This δ is our Lebesgue number:

Suppose B has diameter $< \delta$. Then if $x_0 \in B$, $x_0 \in B \subseteq B_\delta(x_0) \subseteq A_i$, some A_i. $\square$

Now we can prove the uniform continuity theorem, after defining our generalized notion of uniform continuity:

Def: If (X, d_X) and (Y, d_Y) are metric spaces, then a function $f: X \longrightarrow Y$ is <u>uniformly continuous</u> if $\forall \ \varepsilon > 0, \ \exists \ \delta$ s.t. $\forall \ x_0, x_1 \in X, \quad d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \varepsilon$.

Thm: (Uniform continuity theorem) If X and Y are metric spaces, X compact, then any continuous function $f: X \longrightarrow Y$ is uniformly continuous.

Pf: Take $\varepsilon > 0$. Take the open covering of Y by balls of radius $\varepsilon/2$.

Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls.

Let δ be a Lebesgue number for $\mathcal{A}$.

If $x_1, x_2 \in X$ s.t. $d(x_1, x_2) < \delta$, then they both lie in some elt of $\mathcal{A}$, so $f(x_1), f(x_2)$ are in some $\varepsilon/2$ - ball together, so $d(f(x_1), f(x_2)) < \varepsilon$. $\square$

Limit Point and sequential compactness

So far we have two definitions of compactness:
one involving open sets, one just for subsets of $\mathbb{R}^n$ (closed and bded).
We'll introduce two more:

Def: A space X is <u>limit point compact</u> if every infinite subset of X has a limit point.

Ex: 1.) $(0,1]$ is not limit point compact: $\{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ has no limit points.

2.) $\{1/n \mid n \in \mathbb{Z}_+\} \cup \{0\}$ is limit point compact: every infinite subset has 0 as a limit point.

3.) $\mathbb{R}$ is not limit point compact: $\mathbb{Z}$ has no limit points.

For a general topological space, limit point compactness $\not\Rightarrow$ compactness.

Ex: Give $\mathbb{Z}$ the topology generated by the basis $\{\{n, -n\} \mid n \in \mathbb{Z}_+\}$ and $\{0\}$. Call this space X.

Any infinite set contains some $n \in \mathbb{Z}$, so $-n$ is a limit point.
Thus X is limit point compact.

However, the basis is a cover w/ no finite subcover, so X is not Compact.

The converse holds in full generality:

Theorem: Compactness $\Longrightarrow$ limit point compactness.

Pf: Let X be compact. Suppose X is not limit point compact.

Then there is some $A \subseteq X$ infinite w/ no limit point.

Thus A vacuously contains all its limit points $\Rightarrow$ A is closed.

For each $a \in A$, we can find a neighborhood $U_a \ni a$ s.t.

$U_a \cap A = \{a\}$. Then the U_a and $X - A$ give an open cover of X w/ no finite subcover since A is infinite, which contradicts the compactness of X. $\square$

For metric spaces, these two types of compactness are the same, which we will see soon. In fact, there is one more type of compactness equivalent to them both for metric spaces:

Def: X is <u>sequentially compact</u> if every sequence of points of X has a convergent subsequence.

Ex: $-1, 1, -1, 1, -1, \ldots$ has $1, 1, 1, \ldots$ as a convergent subseq.

$\underset{\uparrow}{1}, 1, \underset{\uparrow}{\tfrac{1}{2}}, 2, \underset{\uparrow}{\tfrac{1}{3}}, 3, \underset{\uparrow}{\tfrac{1}{4}}, 4, \ldots$ has a conv. subsequence.

Ex: $1, 2, 3, \ldots$ has no convergent subsequence in $\mathbb{R}$, so $\mathbb{R}$ is not sequentially compact.

In general, sequential compactness $\underset{\not\Rightarrow}{\not\Leftarrow}$ compactness! However...

Thm: If (X, d) is a metric space, w/ induced metric topology, then the following are equivalent:

1.) X is compact

2.) X is limit point compact

3.) X is sequentially compact.

For the proof we need the following:

Lemma: X seq. compact metric space, $\mathcal{A}$ an open cover of X, then $\mathcal{A}$ has a Lebesgue number.

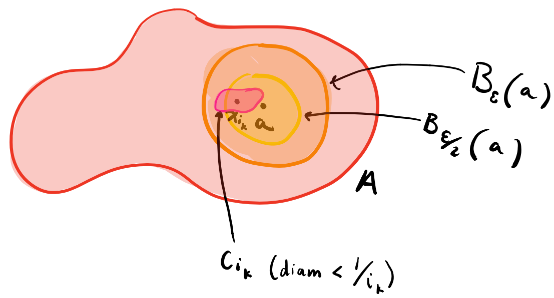
Pf: Let $\mathcal{A}$ be an open cover. Suppose it has no Lebesgue number. i.e. $\forall\ \delta > 0$, $\exists\ C_\delta$ w/ diam. $< \delta$, not in any elt of $\mathcal{A}$.

For each $n \in \mathbb{Z}_+$, pick $x_n \in C_{1/n}$.

There is some convergent subseq. $x_{i_1}, x_{i_2}, \ldots \longrightarrow a$.

$a \in A$, some $A \in \mathcal{A}$, so $a \in B_\varepsilon(a) \subseteq A$, some $\varepsilon > 0$.

Choose $k >> 0$ s.t. $d(x_{i_k}, a) < \varepsilon/2$ and $\frac{1}{i_k} < \varepsilon/2$



But then $C_{i_k} \subseteq A$, a contradiction. $\square$

Pf of Theorem: 1.) $\Rightarrow$ 2.) : done.

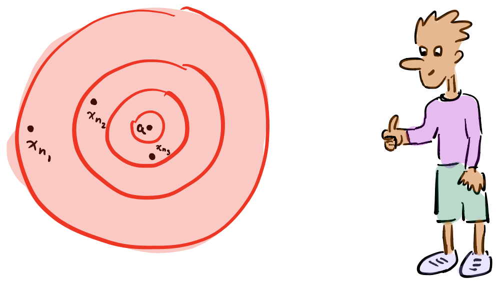
2.) $\Rightarrow$ 3.): Suppose X is limit point compact.

Consider a sequence $x_1, x_2, \ldots$. If $\{x_1, x_2, \ldots\}$ is finite, then $\exists$ $x = x_n$ for infinitely many n, which is thus a convergent subsequence.

If $\{x_1, x_2, \ldots\}$ is infinite, it has a limit point a.

We can find $x_{n_1} \in B_1(a)$. Then for $i > 1$, choose

$x_{n_i} > x_{n_{i-1}}$ s.t. $x_{n_i} \in B_{1/i}(a)$. Then $x_{n_1}, x_{n_2}, \ldots$ is a subseq. that converges to a.



3.) $\Rightarrow$ 1.): Assume X is sequentially compact.

Let $\mathcal{A}$ be an open cover of X, w/ Lebesgue number δ (by lemma).

Set $\varepsilon = \delta/3$. Claim: there is a finite cover of X be ε-balls.

Suppose there's not...

Set $x_1 \in X$, and for $n > 1$, $x_n \in X \setminus \bigcup_{i=1}^{n-1} B_\varepsilon(x_i)$. Then by sequential compactness, some subsequence converges. But this is impossible, since no two points are w/in ε of each other.

Thus, we can take a finite cover by ε-balls, which have diameter $\frac{2\delta}{3} < \delta$, so each one is contained in an element of $\mathcal{A}$.

We then get a finite subcover of $\mathcal{A}$ be replacing each ε-ball w/ the elt of A that contains it! $\square$